2020 A

Lecture 6

Fall 2020/21

- Improper Integral
- Triple Integral

In Riemann integral, the fcns under consideration are bdd and the region of integration is bdd. When the integrand becomes unbdd, or when the region becomes unbdd, we need to define the integral from the beginning. We call it the improper integral.

We consider double integral over $\mathbb{R}^2$ as a special case.

Let f be defined in $\mathbb{R}^2$. Suppose it is integrable in every bdd region. Define

$$\iint_{\mathbb{R}^2} f = \lim_{r\to\infty} \iint_{D_r} f \ , \quad D_r = \{(x,y) : x^2+y^2 \leq r^2\}$$

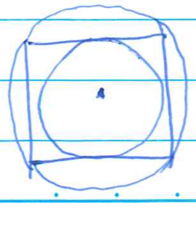
provided the limit exists.

Theorem 13

$$\lim_{r\to\infty} \iint_{D_r} f = \lim_{r\to\infty} \iint_{Q_r} f, \quad Q_r \text{ square of side } r \text{ center at } (0,0).$$

provided either limit exists.

Pf:



$$D_r \subseteq Q_r \subseteq D_{\sqrt{2}r} \subseteq Q_{\sqrt{2}r}$$

$$\iint_{D_r} f \leq \iint_{Q_r} f \leq \iint_{D_{\sqrt{2}r}} f \leq \iint_{Q_{\sqrt{2}r}} f \ , \text{ so done } \#$$

This Theorem tells us that you may use disks or squares to define the improper integral.

e.g. 12 Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

$$\left(\int_{-a}^{a} e^{-x^2}\,dx\right)^2 = \left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$$

$$= \int_{-a}^{a}\int_{-a}^{a} e^{-x^2-y^2}\,dx\,dy$$

$$= \iint_{Q_a} e^{-(x^2+y^2)}.$$

On the other hand,

$$\iint_{D_a} e^{-(x^2+y^2)} = \iint_{D_a} e^{-r^2}$$

$$= \int_0^{2\pi}\int_0^{a} e^{-r^2}\,dr\,d\theta$$

$$= 2\pi\,\frac{1}{2}\left(1-e^{-a^2}\right)$$

$$\therefore \quad \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)} = \pi.$$

By Theorem,

$$\lim_{a\to\infty}\iint_{Q_a} e^{-x^2-y^2} = \lim_{a\to\infty}\iint_{D_a} e^{-r^2} = \pi$$

$\therefore$

$$\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}dx\right)^2 = \lim_{a\to\infty}\iint_{Q_a} e^{-x^2-y^2} = \pi$$

$\therefore$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

For unbdd fcn, we consider a special case.

e.g.13 Evaluate

$$\iint_{D_1} \frac{1}{(x^2+y^2)^{\frac{\alpha}{2}}}, \quad \alpha>0, \quad \text{if possible.}$$

Using polar coor,

$$\iint_{D_1\setminus D_r} \frac{1}{(x^2+y^2)^{\frac{\alpha}{2}}} = \int_0^{2\pi}\int_r^1 \frac{1}{r^\alpha}\, r\,dr\,d\theta$$

$$= 2\pi \left.\frac{r^{2-\alpha}}{2-\alpha}\right|_r^1 \qquad (\alpha\neq 2)$$

$$= \frac{2\pi}{2-\alpha}\left(1-r^{2-\alpha}\right)$$

and, when $\alpha = 2$,

$$\iint\limits_{P_1 \setminus P_r} \frac{1}{(\ )^{\frac{\alpha}{2}}} = 2\pi(1 - \log r)$$

Therefore,

$$\lim_{r \to 0} \iint\limits_{P_1 \setminus P_r} \frac{1}{(x^2+y^2)^{\frac{\alpha}{2}}} = \begin{cases} \frac{2\pi}{2-\alpha}, & \alpha < 2 \\ \infty, & \alpha \geq 2 . \end{cases}$$

We conclude that the improper integral exists iff $\alpha < 2$.

X X X X

Let B be a rectangular box

$$B = [a,b] \times [c,d] \times [e,f]$$

A partition on B, P, is

$$a = x_0 < x_1 < \cdots < x_n = b$$
$$c = y_0 < y_1 < \cdots < y_m = d$$
$$e = z_0 < z_1 < \cdots < z_\ell = f$$

subrectangular box : $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$.

its volume $|B_{ijk}| = \Delta x_i \Delta y_j \Delta z_k$.

Let $p_{ijk} \in B_{ijk}$, we form the Riemann sum of a function F in B w.r.t. $\dot{P}$ as

$$S(F, \mathring{P}) = \sum_{i,j,k} F(p_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

$$\overset{\text{or}}{=} \sum_{i,j,k} F(p_{ijk}) |B_{ijk}| .$$

F is integrable over B if $\exists\, I \in \mathbb{R}$ so that $\forall \varepsilon, \exists \delta$

$$\left| S(F, \mathring{P}) - I \right| < \varepsilon, \quad \forall P, \|P\| < \delta .$$

$\|P\| = \max_{i,j,k} \{ \Delta x_i, \Delta y_j, \Delta z_k \}$. We may also write

$$I = \lim_{\|P\| \to 0} S(F, \mathring{P}) .$$

Notation: Use

$$\iiint_B F \quad , \quad \iiint_B F\, dV \quad \text{to denote } I .$$

Note that same as for double integral

- $\exists$ non-integrable functions
- integrable $\Longrightarrow$ bounded
- continuous $\Longrightarrow$ integrable.

<u>Fubini's theorem</u> Let F be conti in B. Then

$$\iiint_B F = \iint_R \left(\int_e^f F(x,y,z)\,dz \right), \quad R = [a,b]\times[c,d],$$

So, a triple integral is reduced to a single and a double integral.

"Pf". For $\varepsilon > 0$, $\exists\, \delta$ s.t.

$$|F(p) - F(q)| < \varepsilon, \quad \forall\, p, q \in B_{ijk}, \ \|P\| < \delta. \quad (*)$$

($\because$ F is continuous in B.) Pick tag points of the form $(x_i^*, y_j^*, z_k^*) \in B_{ijk}$ and consider

$$S(F, \mathring{P}) = \sum_{i,j} \left(\sum_k F(x_i^*, y_j^*, z_k^*)\,\Delta z_k \right) \Delta x_i \Delta y_j$$

$$= \sum_{i,j} \left(\sum_k \left[F(x_i^*, y_j^*, z_k^*) - \frac{1}{\Delta z_k} \int_{z_{k-1}}^{z_k} F(x_i^*, y_j^*, z)\,dz \right] \Delta z_k \right) \Delta x_i \Delta y_j$$

$$+ \sum_{i,j} \left(\sum_k \int_{z_{k-1}}^{z_k} F(x_i^*, y_j^*, z)\,dz \right) \Delta x_i \Delta y_j. \quad (\star)$$

Let m, M be the min, max of $F(x_i^*, y_j^*, z)$ over $z \in [z_{k-1}, z_k]$

Then $\quad m \le F(x_i^*, y_j^*, z_k^*), \ \frac{1}{\Delta z_k} \int_{z_{k-1}}^{z_k} F(x_i^*, y_j^*, z)\,dz \le M$

By $(*)$, $M - m \le \varepsilon$, so

$$\left| F(x_i^*, y_j^*, z_k^*) - \frac{1}{\Delta z_k} \int_{z_{k-1}}^{z_k} F(x_i^*, y_j^*, z)\,dz \right| \le \varepsilon$$

and the first term in ($\star$)

$$\leq \varepsilon \sum_{i,j} \left(\sum_k \Delta z_k \right) \Delta x_i \Delta y_j$$

$$= \varepsilon (b-a)(d-c)(f-e)$$

Denote this term by E,

$$S(F, \mathring{P}) = E + \sum_{i,j} \sum_k \int_{z_{k-1}}^{z_k} F(x_i^*, y_j^*, z)\, dz\, \Delta x_i \Delta y_j$$

$$= E + \sum_{i,j} \int_e^f F(x_i^*, y_j^*, z)\, dz\, \Delta x_i \Delta y_j$$

Let $G(x,y) = \int_e^f F(x,y,z)\, dz$ be conti in $(x,y) \in R$.

$$= E + \sum_{i,j} G(x_i^*, y_j^*) \Delta x_i \Delta y_j$$

As G is integrable over R, as $\|P\| \to 0$,

$Q: a = x_0 < x_1 < \cdots < x_n = b$
$c = y_0 < y_1 < \cdots < y_m = d$

$$S(F, \mathring{P}) = E + S(G, \mathring{Q}),$$

$$\to 0 + \iint_R G(x,y)\, dA$$

$$= \iint_R \int_e^f F(x,y,z)\, dz\, dA.$$